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Structure of the reflection coefficient and the eigenvalue problem of the Fokker–Planck equation

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Abstract. Structure of the reflection coefficient of the Fokker-Planck equation is investigated. Asymptotic expressions of the reflection coefficient for small wavenumber (low frequency) and large wavenumber (high frequency) are presented. As an application of this analysis, a method for calculating the eigenvalues of the Fokker-Planck equation is derived.

1. Introduction

Diffusion in a one-dimensional potential $U(x)$ is described by the Fokker-Planck equation [1, 2]

$$\frac{\partial}{\partial t}\Phi(x,t)=\frac{\partial^2}{\partial x^2}\Phi(x,t)-2\frac{\partial}{\partial x}[f(x)\Phi(x,t)] \tag{1}$$

where

$$f(x)\equiv-\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}x}U(x). \tag{2}$$

Solutions of (1) show various long time behaviour depending on the asymptotic form of $U(x)$ at infinity. If $U(x)$ diverges faster than $|x|$ as $x\to\pm\infty$, the large-t behaviour of $\Phi(x,t)$ is determined by low-lying eigenvalues of (1). If we set

$$\Phi(x,t)=\mathrm{e}^{-\lambda t}\phi(x) \tag{3}$$

where λ is the eigenvalue, (1) reduces to

$$\frac{\mathrm{d}^2}{\mathrm{d}x^2}\phi-2\frac{\mathrm{d}}{\mathrm{d}x}(f\phi)+\lambda\phi=0. \tag{4}$$

There are several methods for calculating the eigenvalues of the Fokker-Planck equation [3, 4]; for example, variational methods or the WKB method. Most of them are an application of a method for the Schrödinger equation. (It is well known that a Fokker-Planck equation can be transformed into a Schrödinger equation.) However, in fact, the eigenvalue problem of the Fokker-Planck equation can be treated more simply than that of the Schrödinger equation. In this paper we present a method to calculate eigenvalues of the Fokker-Planck equation in a systematic way. It is based on the analysis of the reflection coefficients for (1).

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The reflection coefficients for the interval (a, b) are defined as follows. We consider the potential $\bar{U}(x)$, which is constant outside this interval and coincides with $U(x)$ within;

$$\begin{aligned} \bar{U}(x) &\equiv U(a) \qquad & x \leqslant a \\ &\equiv U(x) \qquad & a \leqslant x \leqslant b \\ &\equiv U(b) \qquad & b \leqslant x. \end{aligned} \tag{5}$$

Outside (a, b), a solution of the equation

$$\frac{\mathrm{d}^2}{\mathrm{d}x^2}\phi - 2\frac{\mathrm{d}}{\mathrm{d}x}(\bar{f}\phi) + p^2\phi = 0 \qquad \bar{f}(x) \equiv -\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}x}\bar{U}(x) \tag{6}$$

can be written as a linear combination of $\exp(\mathrm{i}px)$ and $\exp(-\mathrm{i}px)$. The reflection coefficients R_l, R_r, and the transmission coefficient T are defined as quantities such that there are two solutions of equation (6) which have the following form outside (a, b):

$$\begin{aligned} \phi_1(x; p) &= \mathrm{e}^{ipx} + R_l(a, b; p)\,\mathrm{e}^{-ipx} \qquad & x \leqslant a \\ &= \mathrm{e}^{U(a)-U(b)}T(a, b; p)\,\mathrm{e}^{ipx} \qquad & b \leqslant x \end{aligned} \tag{7a}$$

$$\begin{aligned} \phi_2(x; p) &= \mathrm{e}^{U(b)-U(a)}T(a, b; p)\,\mathrm{e}^{-ipx} \qquad & x \leqslant a \\ &= \mathrm{e}^{-ipx} + R_r(a, b; p)\,\mathrm{e}^{ipx} \qquad & b \leqslant x \end{aligned} \tag{7b}$$

(figure 1). We may also consider the cases where $a = -\infty$ or $b = \infty$. For such cases the transmission coefficient vanishes and only ϕ_2 (if $a = -\infty$) or ϕ_1 (if $b = \infty$) exists. The reflection coefficients R_l and R_r are in general different, while the transmission coefficients T in (7a) and (7b) are identical. In this paper we shall mainly deal with R_r, since R_l can be discussed completely parallel to it.

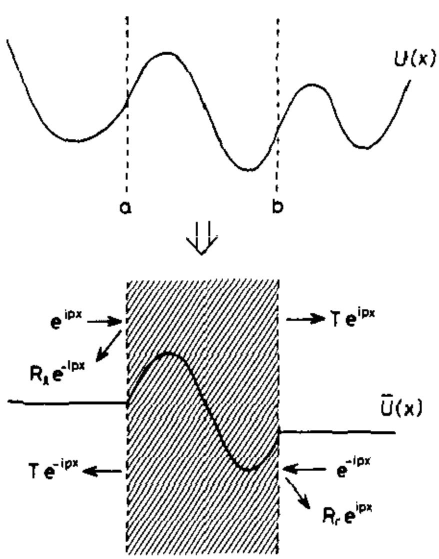

Figure 1. Definition of R_l, R_r and T. (The factors $\exp[U(a) - U(b)]$ and $\exp[U(b) - U(a)]$ are omitted in the figure.)

The analysis of the reflection coefficient is useful not only for the calculation of eigenvalues but also for the study of the asymptotic behaviour of solutions for the cases where the potential $U(x)$ does not give discrete eigenvalues. The frequency component of the Green function of (1) can be expressed in terms of the reflection coefficients and the transmission coefficient [5], where the wavenumber p is replaced by a complex quantity κ satisfying $\kappa^2 = i\omega$, $\mathrm{Im}\,\kappa \geqslant 0$ (ω is the frequency). Hence by investigating the asymptotic behaviour of the reflection coefficients we can gain an insight into the behaviour of the solutions of (1).

2. Reflection coefficient for potentials which diverge at infinity

We use the notation

$$[s_1, s_2, \ldots, s_k]_a^b \equiv \int \ldots \int_{a \leqslant x_1 \leqslant x_2 \leqslant \ldots \leqslant x_k \leqslant b} \mathrm{d}x_1 \ldots \mathrm{d}x_k \exp\left[\sum_{j=1}^{k} s_j U(x_j)\right] \tag{8}$$

where $s_i = \pm 1$ ($i = 1, 2, \ldots$). Results for small p are expressed in terms of integrals of the form (8). For simplicity we also write, for example,

$$[--+-]_a^b \equiv [-1, -1, +1, -1]_a^b$$
$$= \int_a^b \mathrm{d}x_1 \int_{x_1}^b \mathrm{d}x_2 \int_{x_2}^b \mathrm{d}x_3 \int_{x_3}^b \mathrm{d}x_4\, \mathrm{e}^{-U(x_1)-U(x_2)+U(x_3)-U(x_4)}. \tag{9}$$

And we introduce the operators

$$\begin{aligned} J_{+1}(\equiv J_+) &\equiv \mathrm{e}^{U(x)}\left(1 + \frac{\mathrm{d}}{\mathrm{d}U(x)}\right) \\ J_{-1}(\equiv J_-) &\equiv \mathrm{e}^{-U(x)}\left(1 - \frac{\mathrm{d}}{\mathrm{d}U(x)}\right). \end{aligned} \tag{10}$$

Note that these operators, together with $J_z \equiv \mathrm{d}/\mathrm{d}U$, satisfy the same commutation relations as the angular momentum operators;

$$[J_+, J_-] = 2J_z \qquad [J_z, J_+] = J_+ \qquad [J_z, J_-] = -J_-. \tag{11}$$

The expression for $R_r(-\infty, x; p)$ takes a simple form if $U(x)$ goes to $+\infty$ or $-\infty$ faster than $\log|x|$ as $x \to -\infty$. Potentials which have a discrete eigenvalue spectrum satisfy this condition. If $U(x)$ diverges to plus infinity faster than $\log|x|$ in the limit $x \to -\infty$, the reflection coefficient can be expressed as [6]

$$\begin{aligned} R_r(-\infty, x; p) &= 1 + \sum_{n=1}^{\infty} R_n p^n \\ R_1 &= 2\mathrm{i}\, \mathrm{e}^{U(x)}[-1]_{-\infty}^{x} \\ R_n &= \sum_{\{s\}} g_n(s_1, s_2, \ldots, s_{n-1})[-1, s_1, s_2, \ldots, s_{n-1}]_{-\infty}^{x} \qquad (n \geqslant 2) \end{aligned} \tag{12}$$

$$\begin{aligned} g_n(s_1, s_2, \ldots, s_{n-1}) &\equiv 2\mathrm{i}^n J_{-s_{n-1}} J_{-s_{n-2}} \ldots J_{-s_2} J_{-s_1}\, \mathrm{e}^{U(x)} \\ &= 2\mathrm{i}^n \left\{\prod_{j=1}^{n-1}\left[-s_j\left(1 - \sum_{k=1}^{j} s_k\right)\right]\right\} \exp\left[\left(1 - \sum_{i=1}^{n-1} s_i\right) U(x)\right] \end{aligned} \tag{13}$$

where $\Sigma_{\{s\}}$ denotes the sum over $s_1=\pm1,\ldots,s_{n-1}=\pm1$. To order p^5, the expression (12) reads

$$\begin{aligned}
&R_r(-\infty, x; p)\\
&\quad=1+2\mathrm{i}\,\mathrm{e}^{U(x)}[-]_{-\infty}^{x}p-4\,\mathrm{e}^{2U(x)}[--]_{-\infty}^{x}p^2\\
&\qquad-2\mathrm{i}(6\,\mathrm{e}^{3U(x)}[---]_{-\infty}^{x}-2\,\mathrm{e}^{U(x)}[--+]_{-\infty}^{x})p^3\\
&\qquad+2(24\,\mathrm{e}^{4U(x)}[----]_{-\infty}^{x}-12\,\mathrm{e}^{2U(x)}[---+]_{-\infty}^{x}-4\,\mathrm{e}^{2U(x)}[--+-]_{-\infty}^{x})p^4\\
&\qquad+2\mathrm{i}(120\,\mathrm{e}^{5U(x)}[-----]_{-\infty}^{x}-72\,\mathrm{e}^{3U(x)}[----+]_{-\infty}^{x}\\
&\qquad-36\,\mathrm{e}^{3U(x)}[---+-]_{-\infty}^{x}+12\,\mathrm{e}^{U(x)}[---++]_{-\infty}^{x}\\
&\qquad-12\,\mathrm{e}^{3U(x)}[--+--]_{-\infty}^{x}+4\,\mathrm{e}^{U(x)}[--+-+]_{-\infty}^{x})p^5+\ldots.
\end{aligned}\tag{14}$$

The structure of (12) and (13) becomes clear if we express $[-1, s_1, s_2, \ldots, s_{n-1}]$ graphically as shown in figure 2. Apart from the overall factor $2\mathrm{i}^n$ and $\Pi\, s_j$, the coefficient g_n is obtained as the product of the value of the ordinate ($\equiv m$) at each dot, and e^{mU} for the value of m at the end point of the graph. Divergent integrals such as $[--++]_{-\infty}^{x}$ or $[-++-]_{-\infty}^{x}$ do not appear in the expression (12) since the graphs for them touch the $m=0$ line (figure 3).

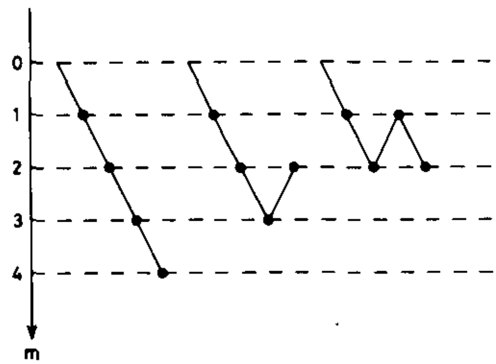

Figure 2. Graphical expression of the three terms in R_4 (the third line in equation (14)). The factors for them are calculated as, respectively, $1\times2\times3\times4=24$, $1\times2\times3\times2=12$ and $1\times2\times1\times2=4$. Since the graphs end at $m=4$, $m=2$ and $m=2$, respectively, we get e^{4U}, e^{2U} and e^{2U}.

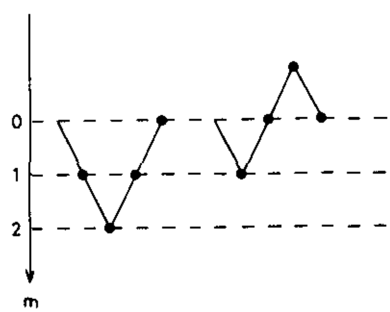

Figure 3. The graphs for $[--++]_{-\infty}^{x}$ and $[-++-]_{-\infty}^{x}$.

Similarly, if $U(x)$ tends to minus infinity faster than $\log|x|$, the reflection coefficient has the form

$$R_r(-\infty, x; p) = -1 + \sum_{n=1}^{\infty} R_n p^n$$

$$R_1 = -2\mathrm{i}\, \mathrm{e}^{-U(x)}[+1]_{-\infty}^{x} \tag{12'}$$

$$R_n = \sum_{\{s\}} g_n(s_1, s_2, \ldots, s_{n-1})[+1, s_1, s_2, \ldots, s_{n-1}]_{-\infty}^{x} \qquad (n \geq 2)$$

$$\begin{aligned} g_n(s_1, s_2, \ldots, s_{n-1}) &\equiv -2\mathrm{i}^n J_{-s_{n-1}} J_{-s_{n-2}} \ldots J_{-s_2} J_{-s_1}\, \mathrm{e}^{-U(x)} \\ &= -2\mathrm{i}^n \left\{ \prod_{j=1}^{n-1} \left[-s_j \left(-1 - \sum_{k=1}^{j} s_k \right) \right] \right\} \exp\left[\left(-1 - \sum_{i=1}^{n-1} s_i \right) U(x) \right]. \end{aligned} \tag{13'}$$

3. Eigenvalue problem

We can derive from (12) the expression (see appendix A)

$$\log R_r(-\infty, x; p) = \mathrm{i} \sum_{n=0}^{\infty} C_{2n+1} p^{2n+1} \tag{15a}$$

$$C_1 = 2\, \mathrm{e}^{U}[-]_{-\infty}^{x} \tag{15b}$$

$$\begin{aligned} C_{2n+1} = \sum_{\{s\}} g_{2n}(s_1, \ldots, s_{2n-1})(&\mathrm{e}^{U(x)}[-1, s_1, s_2, \ldots, s_{2n-1}, -1]_{-\infty}^{x} \\ &- \mathrm{e}^{-U(x)}[-1, s_1, s_2, \ldots, s_{2n-1}, +1]_{-\infty}^{x}) \qquad (n \geq 1) \end{aligned} \tag{15c}$$

where g is given by (13). For instance,

$$\begin{aligned} C_3 &= -4\, \mathrm{e}^{2U(x)}(\mathrm{e}^{U(x)}[---]_{-\infty}^{x} - \mathrm{e}^{-U(x)}[--+]_{-\infty}^{x}) \\ C_5 &= 48 \mathrm{e}^{4U(x)}(\mathrm{e}^{U(x)}[-----]_{-\infty}^{x} - \mathrm{e}^{-U(x)}[----+]_{-\infty}^{x}) \\ &\quad -24\, \mathrm{e}^{2U(x)}(\mathrm{e}^{U(x)}[---+-]_{-\infty}^{x} - \mathrm{e}^{-U(x)}[---++]_{-\infty}^{x}) \\ &\quad -8\, \mathrm{e}^{2U(x)}(\mathrm{e}^{U(x)}[--+--]_{-\infty}^{x} - \mathrm{e}^{-U(x)}[--+-+]_{-\infty}^{x}). \end{aligned} \tag{16}$$

We may use the expression (15) for the calculation of the eigenvalue λ of the Fokker-Planck equation, which satisfies (4). Here we consider the case where $U(x)$ goes to $+\infty$ as $x \to \pm\infty$. Other cases can be similarly treated. Let us denote the nth eigenvalue by λ_n. The lowest eigenvalue is $\lambda_0 = 0$, corresponding to the eigenstate $\phi = \mathrm{e}^{-U}$. If we write $\lambda_n = k_n^2$, the eigenvalue is obtained from the equation [5]

$$R_r(-\infty, x; k_n) R_l(x, \infty; k_n) = 1 \tag{17}$$

where x is arbitrary. This condition is equivalent to

$$\log R_r(-\infty, x; k_n) + \log R_l(x, \infty; k_n) = 2n\pi\mathrm{i} \tag{18}$$

for each n. To calculate the eigenvalues we substitute (15) and the corresponding expression for $\log R_l$,

$$\begin{aligned} &\log R_l(x, \infty; p) = \mathrm{i} \sum_{n=0}^{\infty} C'_{2n+1} p^{2n+1} \\ &C'_1 = 2\, \mathrm{e}^{U}[-]_{x}^{\infty} \\ &C'_3 = -4\, \mathrm{e}^{2U}(\mathrm{e}^{U}[---]_{x}^{\infty} - \mathrm{e}^{-U}[+--]_{x}^{\infty}) \qquad \text{etc} \end{aligned} \tag{19}$$

into (18). We have

$$(C_1+C_1')k_n+(C_3+C_3')k_n^3+\ldots=2n\pi. \tag{20}$$

In particular, if $U(x)$ is a symmetric function of x, equation (20) reduces to

$$C_1k_n+C_3k_n^3+\ldots=n\pi \tag{21}$$

since $C'_{2n+1}=C_{2n+1}$. Suppose that $U(x)$ is a single-well potential which has only one minimum at, say, $x=0$ and rapidly increases with $|x|$. Then the two integrals $e^{U(0)}[-1, s_1, s_2, \ldots, s_{2n-1}, -1]^0_{-\infty}$ and $-e^{-U(0)}[-1, s_1, s_2, \ldots, s_{2n-1}, +1]^0_{-\infty}$, which appear as a pair in the expression of C_{2n+1} for $n\geqslant 1$ (15c) almost cancel one another. Hence the coefficients $C_{2n+1}(n\geqslant 1)$ are small in comparison with C_1. This allows us to retain only the first term in the left-hand side of (20) as the first approximation for λ_1. (We set $x=0$ in (15) and (19).) We get

$$k_1=\frac{2\pi}{C_1+C_1'}=\frac{\pi}{e^{U(0)}\int_{-\infty}^{\infty}e^{-U(x)}\,dx}. \tag{22}$$

For example, the eigenvalue for the potential $U(x)=\frac{1}{4}x^4$ is calculated with (22) as

$$\lambda_1=k_1^2=\left(\frac{\pi}{2\sqrt{2}\,\Gamma(5/4)}\right)^2\simeq 1.50. \tag{23}$$

The precise value obtained by a numerical method is [7] $\lambda_1\simeq 1.37$. This approximation gives a good result for potentials that diverge rapidly as $|x|\to\infty$. It is expected that (22) gives a still better approximation for potentials such as $U=x^6$ or $U=x^8$.

On the other hand, this approximation is not so correct for potentials that diverge more slowly or potentials which have a complicated structure, e.g. double-well potentials. For instance, let us consider $U=x^2$. Since $C_1=C_1'=\sqrt{\pi}$, from (22) we get $\lambda_1=\pi$, while the exact value is $\lambda_1=2$. For such potentials, or when we want to calculate λ_n for large n, we must take into account the higher-order terms in (20) or (21). For the parabolic potential $U=x^2$ the coefficient C_3 can be exactly obtained as

$$C_3=4\left[\frac{\Gamma(3/2)}{4}\log 2-\frac{1}{3!}\left(\frac{\sqrt{\pi}}{2}\right)^3\right]\simeq 0.15. \tag{24}$$

This leads to $\lambda_1\simeq 2.22$.

It is in general difficult to exactly calculate the integrals such as $[--+]^0_{-\infty}$. These integrals can be approximately obtained by expanding them in terms of $[-]^0_{-\infty}$, which is easy to calculate. We expand e^U as

$$e^{U(z)-U(0)}\simeq e^{-U(z)+U(0)}(1+\xi_1[-]^0_z+\xi_2[--]^0_z+\xi_3[---]^0_z+\ldots) \tag{25}$$

where the coefficients $\xi_1, \xi_2 \ldots$ are determined by comparing both sides of (25) order by order as a power series in terms of z. From (25) we get

$$\begin{aligned} &e^{-U(0)}[--+]^0_{-\infty}\\ &\quad=e^{U(0)}([---]^0_{-\infty}+\xi_1[----]^0_{-\infty}+\xi_2[-----]^0_{-\infty}+\ldots)\\ &\quad=e^{U(0)}\left\{\frac{1}{3!}([-]^0_{-\infty})^3+\frac{\xi_1}{4!}([-]^0_{-\infty})^4+\frac{\xi_2}{5!}([-]^0_{-\infty})^5+\ldots\right\}. \end{aligned} \tag{26}$$

The first term in the right-hand side of (26) is cancelled by the first term in the right-hand side of (15c).

As an example, let us calculate C_3 for the potential $U=\frac{1}{4}x^4$:

$$[--+]^0_{-\infty}=\frac{1}{6}([-]^0_{-\infty})^3+\frac{1}{420}([-]^0_{-\infty})^7+\frac{1}{4400}([-]^0_{-\infty})^{11}+\frac{347}{9\,828\,000}([-]^0_{-\infty})^{15}+\dots$$

$$C_1=2[-]^0_{-\infty}=2\sqrt{2}\Gamma(5/4)\simeq 2.56 \tag{27}$$

$$C_3=4\left\{\frac{1}{420}([-]^0_{-\infty})^7+\frac{1}{4400}([-]^0_{-\infty})^{11}+\frac{347}{9\,828\,000}([-]^0_{-\infty})^{15}+\dots\right\}\simeq 0.085.$$

The eigenvalues $\lambda_n=k_n^2$ for $U(x)=\frac{1}{4}x^4$ are obtained from the solutions of $C_1k+C_3k^3=n\pi$ as

$$\lambda_1\simeq 1.37 \qquad \lambda_2\simeq 4.54 \qquad \lambda_3\simeq 8.31 \qquad \lambda_4\simeq 12.2 \qquad \lambda_5\simeq 16.0. \tag{28}$$

This agrees well with the result of a numerical calculation [7]:

$$\lambda_1\simeq 1.37 \qquad \lambda_2\simeq 4.45 \qquad \lambda_3\simeq 8.26 \qquad \lambda_4\simeq 12.8 \qquad \lambda_5\simeq 17.8. \tag{29}$$

Here we do not discuss the convergence of the series. It will be studied elsewhere. Higher-order integrals like $[----+]$ or $[---+-]$ can be similarly calculated.

4. Reflection coefficient for finite intervals

In section 3 we studied the reflection coefficient for the semi-infinite interval $(-\infty, x)$, where the potential $U(x)$ tends to infinity as $x\to-\infty$. In order to study the behaviour of solutions of the Fokker–Planck equation, it is necessary to consider the reflection coefficient for a finite interval (x_0, x). It can be written as [6]

$$R_r(x_0,x;p)=\sum_{n=0}^{\infty}R_np^n \tag{30a}$$

$$R_0=\tanh([U(x_0)-U(x)]/2) \tag{30b}$$

$$R_1=\frac{\mathrm{i}}{2\cosh^2([U(x_0)-U(x)]/2)}(\pm]^x_{x_0} \tag{30c}$$

$$R_n=\sum_{\{s\}}\gamma_n(s_1,s_2,\dots,s_{n-1})(\pm,s_1,s_2,\dots,s_{n-1}]^x_{x_0} \qquad (n\geqslant 2) \tag{30d}$$

$$\begin{aligned}\gamma_n(s_1,s_2,\dots,s_{n-1})\\ &=2\mathrm{i}^nJ_{-s_{n-1}}J_{-s_{n-2}}\dots J_{-s_2}J_{-s_1}\frac{1}{4\cosh^2([U(x_0)-U(x)]/2)}\\ &=2\mathrm{i}^nJ_{-s_{n-1}}J_{-s_{n-2}}\dots J_{-s_2}J_{-s_1}\sum_{m=1}^{\infty}(-1)^{m+1}m\,\mathrm{e}^{m[U(x)-U(x_0)]}\\ &=2\mathrm{i}^n\sum_{m=1}^{\infty}(-1)^{m+1}m\left\{\prod_{j=1}^{n-1}\left[-s_j\left(m-\sum_{k=1}^{j}s_k\right)\right]\right\}\exp\left[\left(m-\sum_{i=1}^{n-1}s_i\right)U(x)\right]\end{aligned} \tag{31}$$

where we have introduced the notation

$$(\pm]_a^b \equiv \int_a^b dz(e^{U(a)-U(z)} - e^{-U(a)+U(z)})$$

$$(\pm, s_1, s_2, \ldots, s_k]_a^b$$

$$\equiv \int\int \ldots \int_{a \leqslant z \leqslant x_1 \leqslant x_2 \leqslant \ldots \leqslant x_k \leqslant b} dz\, dx_1 \ldots dx_k$$

$$\times (e^{U(a)-U(z)} - e^{-U(a)+U(z)}) \exp\left[\sum_{j=1}^{k} s_j U(x_j)\right]. \tag{32}$$

The expression (30) with (31) can be used also for the case $x_0 = -\infty$ or $x = +\infty$, as long as the potential converges to a finite limit sufficiently fast that the integrals of the form (32) are finite.

The expression for γ_n is given as an infinite series. (Compare with (12) and (13).) It is illustrated graphically as figure 4. A finite expression for γ_n can be obtained as follows. We define

$$\tilde{J}_- \equiv e^{-U(x)} \tag{33}$$

and we express $J_{-s_{n-1}} J_{-s_{n-2}} \ldots J_{-s_2} J_{-s_1}$ as

$$\left[\prod_{i=1}^{n-1} (-s_i)\right] J_{-s_{n-1}} J_{-s_{n-2}} \ldots J_{-s_2} J_{-s_1} J_+ = \sum_{k=0}^{m} \eta_k \tilde{J}_-^{m-k} J_+^n \tilde{J}_-^k \tag{34}$$

where

$$m \equiv n - 1 + \sum_{i=1}^{n-1} s_i. \tag{35}$$

The right-hand side of (34) can be obtained by substituting

$$J_- = -J_+ \tilde{J}_- \tilde{J}_- \tag{36}$$

into the left-hand side and using the commutation relation

$$[\tilde{J}_-, J_+] = 1. \tag{37}$$

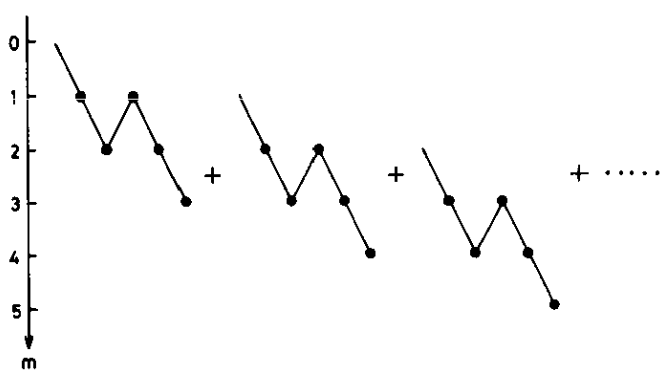

Figure 4. Graphical expression of $(\pm - + - -]_{-\infty}^{x}$. Note that the '$\pm$' plays the role of a '$-$' in the graph.

Using η we may write

$$\gamma_n(s_1, s_2, \ldots, s_{n-1})$$
$$= \frac{2\mathrm{i}^n n! \prod_{i=1}^{n-1}(-s_i)}{2^{n+1}\cosh^{n+1}([U(x_0)-U(x)]/2)}$$
$$\times \exp\left(\frac{1-n}{2}[U(x)-U(x_0)]\right)\exp\left[-\left(\sum_{i=1}^{n-1} s_i\right)U(x)\right]$$
$$\times \sum_{k=0}^{m}(-1)^k \eta_k\, \mathrm{e}^{(k-1)[U(x)-U(x_0)]}. \tag{38}$$

For example, let us calculate R_3. Expression (34) reads

$$\begin{aligned}
J_+J_+J_+ &= J_+J_+J_+\\
-J_-J_+J_+ &= \tfrac{1}{3}\tilde{J}_-\tilde{J}_-J_+J_+J_+ + \tfrac{2}{3}\tilde{J}_-J_+J_+J_+\tilde{J}_-\\
-J_+J_-J_+ &= \tfrac{2}{3}\tilde{J}_-J_+J_+J_+\tilde{J}_- + \tfrac{1}{3}J_+J_+J_+\tilde{J}_-\tilde{J}_-\\
J_-J_-J_+ &= \tilde{J}_-\tilde{J}_-J_+J_+J_+\tilde{J}_-\tilde{J}_-.
\end{aligned} \tag{39}$$

Hence

$$R_3 = \frac{-3\mathrm{i}}{4\cosh^4([U(x_0)-U(x)]/2)}$$
$$\times\{\mathrm{e}^{U(x_0)+U(x)}(\pm--]_{-\infty}^{x} + (-\tfrac{1}{3}\mathrm{e}^{U(x_0)-U(x)}+\tfrac{2}{3})(\pm-+]_{-\infty}^{x}$$
$$+(\tfrac{2}{3}-\tfrac{1}{3}\mathrm{e}^{-U(x_0)+U(x)})(\pm+-]_{-\infty}^{x} + \mathrm{e}^{-U(x_0)-U(x)}(\pm++]_{-\infty}^{x}\}. \tag{40}$$

5. Expressions for large p

Finally let us analyse the structure of the reflection coefficient for large p. The large-p expressions are directly related to the short-time behaviour of the solutions, and they are useful for many problems, e.g. diffusion in a random medium. Moreover, there is a remarkable similarity between the large-p and the small-p expressions.

We define, in a similar way as (8), the integrals

$$\langle s_1, s_2, \ldots, s_k\rangle_a^b \equiv \int \cdots \int_{a\le x_1\le x_2\le\ldots\le x_k\le b} \mathrm{d}x_1\ldots\mathrm{d}x_k\left(\prod_{i=1}^{k} f(x_i)\right)\exp\left(2\mathrm{i}\sum_{j=1}^{k} s_j p x_j\right). \tag{41}$$

The reflection coefficient may be written (see appendix B)

$$\begin{aligned}
R_r(x_0, x; p) &= \mathrm{e}^{2\mathrm{i}px}\sum_{n=0}^{\infty}\hat{R}_{2n+1}(x_0, x; p)\\
\hat{R}_{2n+1} &\equiv \sum_{\{s_k=\pm1:\sum_{k=1}^{2n} s_k=0\}} g_{2n+1}(s_1, s_2, \ldots, s_{2n})\langle -1, s_1, s_2, \ldots, s_{2n}\rangle_{x_0}^{x}
\end{aligned} \tag{42}$$

where g is the same as (13). Explicitly, this reads

$$R_r = \mathrm{e}^{2\mathrm{i}px}(\langle -\rangle_{x_0}^{x} - 2\langle --+\rangle_{x_0}^{x} + 4\langle --+-+\rangle_{x_0}^{x} + 12\langle ---++\rangle_{x_0}^{x} + \ldots). \tag{43}$$

Higher-order terms in the expansion (42) are smaller when p has large positive imaginary part. So (42) can be used for the study of the large frequency behaviour of the reflection coefficient $R_r(x_0, x; \kappa)$. This expression (42) is valid also for $x_0 = -\infty$ or $x = \infty$. We can generalize (42) and get the expression for powers of R,

$$[R_r(x_0, x; p)]^m = e^{2impx} \sum_{n=0}^{\infty} \hat{R}^m_{2n+m}(x_0, x; p) \tag{44a}$$

$$\hat{R}^m_{2n+m} \equiv \sum_{\{s_k=\pm 1: \Sigma_{k=1}^{2n+m-1} s_k = 1-m\}} g_{2n+m}(s_1, s_2, \ldots, s_{2n+m-1}) \langle -1, s_1, s_2, \ldots, s_{2n+m-1} \rangle^x_{x_0} \tag{44b}$$

The similarity between the large-p expression and the small-p expression is clearly seen if we write

$$\begin{aligned} \frac{1 + i\, e^{-2ipx} R_r(x_0, x; p)}{1 - i\, e^{-2ipx} R_r(x_0, x; p)} &= 1 + 2 \sum_{j=1}^{\infty} [i\, e^{2ipx} R_r(x_0, x; p)]^j \\ &= \sum_{\{s\}} g_n(s_1, s_2, \ldots, s_{n-1}) \langle -1, s_1, s_2, \ldots, s_{n-1} \rangle^x_{x_0}. \end{aligned} \tag{45}$$

The right-hand side of (45) has the same form as (12), where $[-1, s_1, s_2, \ldots, s_{n-1}]p^n$ is replaced by $\langle -1, s_1, s_2, \ldots, s_{n-1} \rangle$. And, similarly, we have

$$\begin{aligned} i\, e^{-U(x)} \frac{1 - R_r(-\infty, x; p)}{1 + R_r(-\infty, x; p)} &= \sum_{n=0}^{\infty} \sum_{\{s_k=\pm 1: \Sigma_{k=1}^{2n} s_k = 0\}} g_{2n+1}(s_1, s_2, \ldots, s_{2n}) [-1, s_1, s_2, \ldots, s_{2n}]^x_{-\infty} p^{2n+1} \\ &= [-]^x_{-\infty} p - 2[--+]^x_{-\infty} p^3 \\ &\quad + (4[--+-+]^x_{-\infty} + 12[---++]^x_{-\infty}) p^5 + \ldots \end{aligned} \tag{46}$$

where the right-hand side has the same structure as (42).

6. Conclusion

We have studied the properties of the reflection coefficient of the Fokker–Planck equation, and derived some expressions for small and large wavenumbers. As an application we have considered only the eigenvalue problem, but the results of this paper are applicable to many other problems. Especially they are essential for the analysis the asymptotic behaviour of solutions, and they are also closely related to the theory of integrable systems.

As can be seen from the results of this paper, the reflection coefficient has an interesting algebraic structure. It is expected that further investigation from an algebraic or geometrical viewpoint will produce results which are practically useful as well as theoretically interesting.

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Appendix 1

The reflection coefficient satisfies the differential equation [6]

$$\frac{\mathrm{d}}{\mathrm{d}x} R_r(-\infty, x; p) = 2\mathrm{i}pR_r(-\infty, x; p) + f(x)[1 - R_r^2(-\infty, x; p)]. \tag{A1.1}$$

From (A1.1) it follows that

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}x} \log R_r(-\infty, x; p) \\ &= 2\mathrm{i}p + f(x)\left[\frac{1}{R_r(-\infty, x; p)} - R_r(-\infty, x; p)\right] \\ &= 2\mathrm{i}p + f(x)[R_r(-\infty, x; -p) - R_r(-\infty, x; p)]\end{aligned} \tag{A1.2}$$

where we have used the relation

$$R_r(-\infty, x; p)R_r(-\infty, x; -p) = 1. \tag{A1.3}$$

Substituting (15) and (12) into (A1.2) we get

$$\mathrm{i}\frac{\mathrm{d}}{\mathrm{d}x}\left(\sum_{n=0}^{\infty} C_{2n+1}p^{2n+1}\right) = 2\mathrm{i}p - 2f \sum_{n=0}^{\infty} R_{2n+1}p^{2n+1} \tag{A1.4}$$

which gives

$$\frac{\mathrm{d}}{\mathrm{d}x} C_1 = 2 + 2\mathrm{i}fR_1 = 2 - 4f(x)\,\mathrm{e}^{U(x)}[-]_{-\infty}^{x} \tag{A1.5}$$

and, for $n \geqslant 1$,

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}x} C_{2n+1} = 2\mathrm{i}fR_{2n+1} \\ &= 2\mathrm{i}f \sum_{\{s_1,\ldots,s_{2n},s_{2n+1}\}} g_{2n+1}(s_1, \ldots, s_{2n-1}, s_{2n})[-1, s_1, \ldots, s_{2n-1}, s_{2n}]_{-\infty}^{x} \\ &= 2\mathrm{i}f \sum_{\{s_1,\ldots,s_{2n}\}} \{g_{2n+1}(s_1, \ldots, s_{2n-1}, -1)[-1, s_1, \ldots, s_{2n-1}, -1]_{-\infty}^{x} \\ &\quad + g_{2n+1}(s_1, \ldots, s_{2n-1}, +1)[-1, s_1, \ldots, s_{2n-1}, +1]_{-\infty}^{x}\}.\end{aligned} \tag{A1.6}$$

From (13) we know that

$$g_{2n+1}(s_1, \ldots, s_{2n-1}, s_{2n}) = -\mathrm{i}s_{2n}\left(1 - \sum_{k=1}^{2n} s_k\right)\mathrm{e}^{-s_{2n}U} g_{2n}(s_1, \ldots, s_{2n-1}). \tag{A1.7}$$

So (A1.6) can be written as

$$\frac{d}{dx} C_{2n+1} = -2if(x) \sum_{\{s_1,\ldots,s_{2n}\}} g_{2n}(s_1,\ldots,s_{2n-1})$$
$$\times\left[\left(1-\sum_{k=1}^{2n-1} s_k+1\right) e^{U(x)}[-1, s_1,\ldots,s_{2n-1},-1]_{-\infty}^{x}\right.$$
$$\left.-\left(1-\sum_{k=1}^{2n-1} s_k-1\right) e^{-U(x)}[-1, s_1,\ldots,s_{2n-1},+1]_{-\infty}^{x}\right] \qquad (n \geqslant 1). \tag{A1.8}$$

Using the relations

$$\frac{d}{dx} g_{2n}(s_1,\ldots,s_{2n-1}) = -2f\left(1-\sum_{i=1}^{2n-1} s_i\right) g_{2n}(s_1,\ldots,s_{2n-1}) \tag{A1.9}$$

and

$$\frac{d}{dx}[-1, s_1,\ldots,s_{2n-1}, s_{2n}]_{-\infty}^{x} = e^{s_{2n}U(x)}[-1, s_1,\ldots,s_{2n-1}]_{-\infty}^{x} \tag{A1.10}$$

it can be easily verified that C_1 and C_{2n+1} $(n \geqslant 1)$ given by (15*b*) and (15*c*) satisfy (A1.5) and (A1.8), respectively.

Appendix 2

Equation (A1.1) transforms into

$$\frac{d}{dx}(e^{-2ipx}R_r)^m = mf[e^{-2ipx}(e^{-2ipx}R_r)^{m-1} - e^{2ipx}(e^{-2ipx}R_r)^{m+1}] \tag{A2.1}$$

for $n = 1, 2, \ldots$. Substituting (44*a*) into (A2.1) leads to

$$\frac{d}{dx}(\hat{R}_r^m)^{(2n+m)} = mf[e^{-2ipx}(\hat{R}_r^{m-1})^{(2n+m-1)} - e^{2ipx}(\hat{R}_r^{m+1})^{(2n+m-1)}]. \tag{A2.2}$$

This means that each term in the expression of $(\hat{R}_r^m)^{(l)}$, which has the form $C\langle s_1,\ldots,s_l\rangle_a^b$, gives rise to the terms $(m+1)C\langle s_1,\ldots,s_l,-1\rangle_a^b$ and $-(m-1)\times C\langle s_1,\ldots,s_l,+1\rangle_a^b$ in $(\hat{R}_r^{m+1})^{(l+1)}$ and $(\hat{R}_r^{m-1})^{(l+1)}$, respectively. Since $\hat{R}_r^{(1)} = \langle - \rangle_a^b$, we get (44*b*).

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